ME 4555 — Linearity

Most of the systems we will study in this class are "linear and time-invariant" (LTI). They have the following 4 properties (assuming the system starts at rest; all states are zero)

1) <u>homogeneity</u> : scaling the input scales the output.

if $u(t) \to \boxed{G} \to y(t)$

then $k\,u(t) \to \boxed{G} \to k\,y(t)$

2) <u>additivity</u> : adding inputs adds outputs.

if $u_1(t) \to \boxed{G} \to y_1(t)$

and $u_2(t) \to \boxed{G} \to y_2(t)$

then $u_1(t)+u_2(t) \to \boxed{G} \to y_1(t)+y_2(t)$

these two properties together are called "superposition".

A system that satisfies superposition is called <u>linear</u>

3) <u>time-invariance</u> : delaying input delays output

if $u(t) \to \boxed{G} \to y(t)$

then $u(t-\tau) \to \boxed{G} \to y(t-\tau)$ for all τ.

4) <u>causality</u> : future can't affect the past. (all "real" systems are causal)

if $u(t) \to \boxed{G} \to y(t)$ and we change $u(t)$ for $t \geq T$ then only $y(t)$ for $t \geq T$ can be affected.

<u>NOTE</u>: Strictly speaking, an LTI system is defined as satisfying properties 1, 2, 3. However, all systems we will study are also causal, so we include property 4 as well.

An LTI system is a generalization of the concept of a linear equation. For example, if $u_1, \dots, u_m$ are inputs and $y_1, \dots, y_m$ are outputs, then the linear equations:

$$y = Au \quad , \quad \text{i.e.}: \left\{ \begin{array}{l} y_1 = a_{11} u_1 + a_{12} u_2 + \cdots + a_{1n} u_n \\ y_2 = a_{21} u_1 + a_{22} u_2 + \cdots + a_{2n} u_n \\ \vdots \qquad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ y_m = a_{m1} u_1 + a_{m2} u_2 + \cdots + a_{mn} u_n \end{array} \right\} \quad \text{are an LTI system.}$$

Derivatives and integrals, i.e. $y = \frac{d}{dt} u$ and $y = \int_0^t u(\tau)\, d\tau$ are also LTI.

In general, any system of ordinary differential equations where each term is of the form $\underbrace{a}_{\text{constant}} \cdot \underbrace{x^{(k)}(t)}_{k^{th} \text{ derivative of } x(t) \text{ or any other signal}}$ and they are added together is an LTI. It could also contain integrals but typically we differentiate such equations so that all the integrals disappear.

Examples:

$m\ddot{x} + c\dot{x} + kx = f$ (with input f and output x) is LTI.

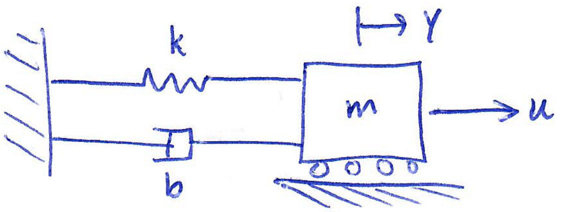
$ml^2\ddot{\theta} + mgl\sin\theta = T$ (with input T and output θ) is not LTI ($\sin\theta$ is not linear)

$m\ddot{x} + \underbrace{ct^2}_{\text{damping increases over time}}\dot{x} + kx = f$ (input f and output x) is not LTI (not time-invariant)

$ml^2\ddot{\theta} + mgl\theta = T$ (small-angle approximation of pendulum) is LTI

Terms like $\dot{\theta}^2$, $\sin\theta$, $\theta\dot{\theta}$, $x_1 x_2$, etc. are all nonlinear.

Proof of linearity for spring-mass-damper system:



$\begin{cases} \text{input: force } u(t) \\ \text{output: position } y(t) \end{cases}$

$$\text{EOM:} \quad m\ddot{y} + b\dot{y} + ky = u.$$

Homogeneity: Suppose we use input $u(t)$ and obtain output $y(t)$. Then $m\ddot{y} + b\dot{y} + ky = u$. Multiply both sides by any $\alpha \in \mathbb{R}$.

$\Rightarrow \alpha(m\ddot{y} + b\dot{y} + ky) = \alpha u \Rightarrow m(\alpha\ddot{y}) + b(\alpha\dot{y}) + k(\alpha y) = \alpha u.$

Since $\alpha \frac{d}{dt}(y(t)) = \frac{d}{dt}(\alpha y(t))$, this means $(\alpha u(t), \alpha y(t))$ also satisfies the same ODE.

Additivity: Suppose (u_1, y_1) and (u_2, y_2) satisfy the ODE. Then $m\ddot{y}_1 + b\dot{y}_1 + ky_1 = u_1$ and $m\ddot{y}_2 + b\dot{y}_2 + ky_2 = u_2$.

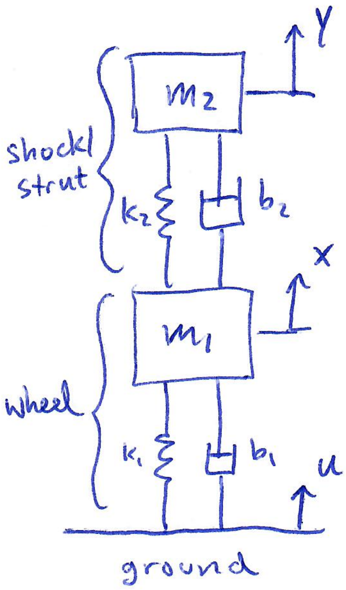
Adding them together: $m(\ddot{y}_1 + \ddot{y}_2) + b(\dot{y}_1 + \dot{y}_2) + k(y_1 + y_2) = (u_1 + u_2)$

since $\frac{d}{dt}(y_1(t)) + \frac{d}{dt}(y_2(t)) = \frac{d}{dt}(y_1(t) + y_2(t))$, this means that $(u_1 + u_2, y_1 + y_2)$ satisfies the same ODE.

NOTE: $y = ku + c$ is not linear! To see why, suppose (u, y) is a solution: so $y = ku + c$.

If this were linear, then $(\alpha u, \alpha y)$ would also be a solution. So we should have $\alpha y = k\alpha u + c$. But instead, we have $\alpha y = \alpha(ku + c) = k\alpha u + \alpha c$. And in general, $\alpha c \neq c$ unless $c = 0$. FOR ANY LINEAR SYSTEM, using $u = 0$ should produce $y = 0$. So constant terms are not allowed

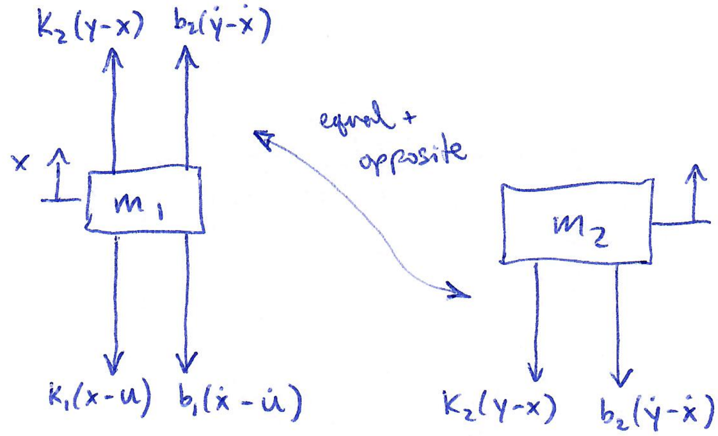
Quarter-car model:



u = position of road [input]

x = c.m. of wheel

y = top of shock/strut assembly [output]



EOM:

$$m_1\ddot{x} = k_2(y-x) + b_2(\dot{y}-\dot{x}) - k_1(x-u) - b_1(\dot{x}-\dot{u})$$

$$m_2\ddot{y} = -k_2(y-x) - b_2(\dot{y}-\dot{x})$$

Rearranged:

$$m_1\ddot{x} + (b_1+b_2)\dot{x} + (k_1+k_2)x - b_2\dot{y} - k_2 y = b_1\dot{u} + k_1 u$$

$$m_2\ddot{y} + b_2\dot{y} + k_2 y - b_2\dot{x} - k_2 x = 0$$

Notes:

1) this system's ODEs contain input derivatives!

2) gravity does not appear because we are measuring x and y with respect to the equilibrium point that already accounts for gravity.

3) since every term is a constant multiple of a derivative, (with no cross-terms like xy, $\dot{x}y^2$, etc.), this system (with u as input, y as output) is LTI.

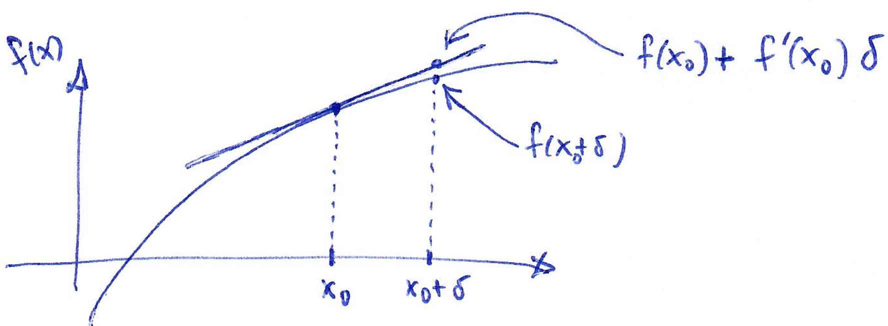
Linearization

Even if a system is nonlinear, we can often approximate its behavior as that of a linear system, provided the state and input stay within some small neighborhood of a given point. We call this point the setpoint (or equilibrium point)

The idea is to use Taylor's theorem. If f is a continuously differentiable function then we can write

$$f(x_0+\delta) = f(x_0) + f'(x_0)\cdot\delta + (\text{higher order terms})$$

Another way to say this is that x_0 is our setpoint and near x_0 (δ is small), we can replace f by a linear function:



For a function of 2 variables $f(x,u)$, we can linearize about any point (x_0, u_0) in an analogous manner:

$$f(x_0+\delta x,\, u_0+\delta u) = f(x_0,u_0) + \frac{\partial f}{\partial x}(x_0,u_0)\cdot\delta x + \frac{\partial f}{\partial u}(x_0,u_0)\,\delta u + \text{H.O.T.}$$

We can apply this idea to a nonlinear system to obtain a linear approximation.

Linearization procedure : For the equation $\dot{x} = f(x,u)$.

1) Pick a setpoint (x_0, u_0). This should satisfy the ODE. So we must have $f(x_0, u_0) = 0$

2) Let $x = x_0 + \delta x$ and $u = u_0 + \delta u$, and replace f by its Taylor approximation:

$$\dot{x} = f(x,u)$$

$$\Rightarrow \quad \frac{d}{dt}(x_0 + \delta x) = f(x_0 + \delta x, u_0 + \delta u)$$

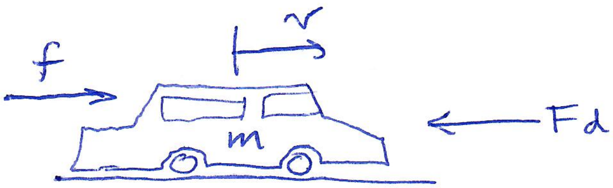
$$\Rightarrow \quad \delta \dot{x} = \underbrace{\cancel{f(x_0, u_0)}}_{\text{zero by choice.}} + \frac{\partial f}{\partial x}(x_0, u_0)\,\delta x + \frac{\partial f}{\partial u}(x_0, u_0)\,\delta u$$

$$\Rightarrow \quad \delta \dot{x} = \underbrace{\left(\frac{\partial f}{\partial x}(x_0, u_0)\right)}_{\text{constant}} \delta x + \underbrace{\left(\frac{\partial f}{\partial u}(x_0, u_0)\right)}_{\text{constant.}} \delta u$$

<u>NOTE</u>: it is also possible to linearize about a time-varying trajectory $(x_0(t), u_0(t))$, which is necessary when the system is time-varying, or we are interested in for example deviations about a given orbit for a satellite.

This also works for systems with higher-order derivatives or multiple inputs and outputs. We will only consider simple cases such as the one above.

Example: cruise control.



f: input force

v: velocity of car

F_d: drag force

EOM: $m\dot{v} = f - F_d$

Assume aerodynamic drag: $F_d = C_d \cdot v^2$ ← drag coefficient (typically depends on air density, area cross-section, ...)

so the EOM is $m\dot{v} + C_d v^2 = f$.

This is nonlinear. Suppose we will be operating at a nominal cruise speed of v_0. Apply Taylor's theorem:

$$v^2 = (v_0 + \delta v)^2 = v_0^2 + 2v_0\,\delta v + \text{H.O.T.}$$

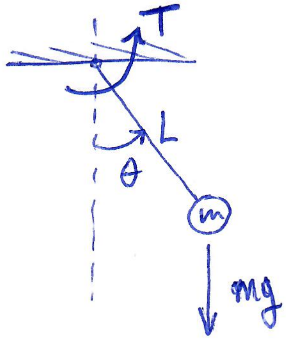
To maintain this speed, we should use $f_0 = C_d v_0^2$.
The linearized EOM is:

$$m\,\delta\dot{v} + (2C_d v_0)\,\delta v = \delta f.$$

Where we can convert back to the original quantities via:

$$\begin{cases} v(t) = v_0 + \delta v(t) \\ f(t) = C_d v_0^2 + \delta f(t). \end{cases}$$

Ex: pendulum.



EOM: $\boxed{mL^2\ddot{\theta} + mgL\sin\theta = T}$

we will linearize this equation about a few different setpoints.

Equilibrium (stable): $\boxed{\theta_0 = 0.}$

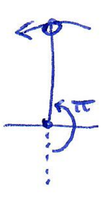
Plugging this in, we find $\boxed{T_0 = 0.}$ Applying Taylor:

$$\sin(\theta_0 + \delta\theta) = \sin(\theta_0) + \cos(\theta_0)\,\delta\theta + \text{H.O.T.} \qquad (*).$$

Let $\theta_0 = 0 \Rightarrow \sin(\delta\theta) \approx \delta\theta$ (this is the small-angle approximation!)

linearized EOM: $\boxed{mL^2\delta\ddot{\theta} + mgL\,\delta\theta = \delta T}$

Equilibrium (unstable): $\boxed{\theta_0 = \pi.}$



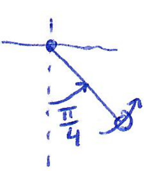
Plugging this in, we find $\boxed{T_0 = 0.}$ Applying Taylor,

Let $\theta_0 = \pi$ in $(*)$ and obtain $\sin(\pi + \delta\theta) \approx \cos(\pi)\,\delta\theta = -\delta\theta.$

linearized EOM: $\boxed{mL^2\delta\ddot{\theta} - mgL\,\delta\theta = \delta T}$

Equilibrium at 45°: $\boxed{\theta_0 = \pi/4}$

Plugging this in, we find $\boxed{T_0 = \frac{\sqrt{2}}{2}mgL}$. Applying Taylor:



Let $\theta_0 = \frac{\pi}{4}$ in $(*)$: $\sin\left(\frac{\pi}{4} + \delta\theta\right) \approx \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\delta\theta$

$$\Rightarrow \sin\left(\frac{\pi}{4} + \delta\theta\right) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\delta\theta$$

linearized EOM: $\boxed{mL^2\delta\ddot{\theta} + \frac{\sqrt{2}}{2}mgL\,\delta\theta = \delta T}$

In this case, δT is the deviation from the nominal torque $T_0 = \frac{\sqrt{2}}{2}mgL$, which is the torque needed to keep the